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$L^{p(\cdot)}$ A PRIORI ESTIMATES FOR POISSON EQUATION

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Let Ω be a bounded domain in $\mathbf{R}^n$ *with* $\partial\Omega \in C^2$ *and let u be a solution of the classical Poisson problem in Ω; i.e.,*

$$\begin{cases} -\Delta u = f & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

where $f \in L^{p(\cdot)}(\Omega)$, $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$.

The main goal of this paper is to prove the following a priori estimate

$$\|u\|_{W^{2,p(\cdot)}(\Omega)} \le C\|f\|_{L^{p(\cdot)}(\Omega)},$$

Key words: Poisson equation, Green function, Calder_on-Zygmund theory, Lebesgue spaces with variable exponents, Sobolev spaces with variable exponents.

1. Introduction

The investigation of the Lebesgue spaces $L^{p(\cdot)}(\Omega)$ with variable exponent was initiated in [25]. Various mathematical problems with variable exponents have been investigated by many authors in recent years. We refer to the overview papers [10, 15, 24] for the advances and the references of this area, and to the monograph [23] for the application background.

We will use the standard notation for Sobolev spaces and for derivatives, namely, if α is a multi-index, $\alpha = (\alpha_1; \alpha_2, ..., \alpha_n) \in \mathbf{Z}_+^n$ we denote

$$|\alpha| = \sum_{j=1}^{n} \alpha_j,\ D^\alpha = \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n} \text{ and}$$

$$W^{k,p(\cdot)}(\Omega) = \{v \in L^{p(\cdot)}(\Omega) : D^\alpha v \in L^{p(\cdot)}(\Omega),\ \forall |\alpha| \le k\}.$$

Let Γ be the standard fundamental solution of the Laplacian operator, namely,

$$\Gamma(x)=\begin{cases}\frac{1}{2\pi}\log|x|^{-1} & n=2\\ \frac{1}{n(n-2)\omega_n}|x|^{2-n} & n\geq 3\end{cases}$$

with ω_n the area of the unit sphere in $\mathbf{R}^n$.

Given a function $f\in C_0^\infty(\mathbf{R}^n)$, it is a classic result that the potential u given by

$$u(x)=\int\Gamma(x-y)f(y)dy$$

is a solution of $-\Delta u=f$ in $\mathbf{R}^n$ and satisfies the estimate

$$\|u\|_{W^{2,p(\cdot)}(\mathrm{R}^n)}\leq C\|f\|_{L^{p(\cdot)}(\mathrm{R}^n)}, \tag{1.1}$$

for $1 < p < \infty$. Indeed, this estimate is a consequence of the Calderón-Zygmund theory of singular integrals (see for example [26]).

Since the work by Komori and Shirai [16], many results on weighted Morrey estimates for maximal functions and singular integral operators have been obtained. In particular, generalizations of (1.1) to weighted Morrey norms are known to hold for weights in the class Ap (see for example [27]).

On the other hand, a priori estimates like (1.1) for solutions of the Dirichlet problem {

$$\begin{cases}-\Delta u=f \ \ in \ \ \Omega\\ u=0 \quad on \ \ \partial\Omega,\end{cases} \tag{1.2}$$

on smooth bounded domains Ω are also well known (see for example the classic paper by Agmon, Douglis and Nirenberg [3] where a priori estimates for general elliptic problems are proved).

Therefore, it is a natural question whether weighted a priori estimates are valid also for the solution of the Dirichlet problem (1.2). In this paper we give a positive answer to this question, namely, we prove that

$$\|u\|_{W^{2,p(\cdot)}(\Omega)}\leq C\|f\|_{L^{p(\cdot)}(\Omega)},$$

for $\omega\in A_p$, $1<p<\infty$, $0\leq k<1$, where the constant C depends only on Ω and on the weight ω.

As an application we obtain weighted Morrey a priori estimates for weights given by powers of the distance to $\partial\Omega$. Estimates of this type are of interest in the analysis of some non-linear problems and were derived using different arguments (see [28]).

2. Preliminaries on variable exponent Lebesgue spaces

Let $p(\cdot)$ be a measurable function on Ω with values in $[1,\infty)$. An open set Ω is assumed to be bounded throughout the whole paper. We suppose that

$$1<p_-\leq p(x)\leq p_+<\infty, \tag{2.1}$$

where $p_- := \operatorname{ess\,inf}_{x\in\Omega} p(x) > 1$; $p_+ := \operatorname{ess\,inf}_{x\in\Omega} p(x) < \infty$.

By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(\Omega)(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : I_{p(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\},$$

this is a Banach function space. By $p'(\cdot) = \dfrac{p(x)}{p(x)-1}$, $x \in \Omega$; we denote the conjugate exponent. The Hölder inequality is valid in the form

$$\int_\Omega |f(x)|\,|g(x)|dx \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}$$

For the basics on variable exponent Lebesgue spaces we refer to [25], [17].

Definition 2.1. *By* $WL(\Omega)$ *(weak Lipschitz) we denote the class of functions defined on* Ω *satisfying the log-condition*

$$|p(x) - p(y)| \le \frac{A}{-\ln|x-y|}, \quad |x-y| \le \frac{1}{2} \; x, y \in \Omega, \tag{2.2}$$

where $A = A(p) > 0$ *does not depend on* x, y.

Let $f \in L_1^{loc}(\mathbb{R}^n)$. The maximal operator M is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy,$$

where $|B(x, t)|$ is the Lebesgue measure of the ball $B(x, t)$.

Let T be a singular integral Calderon-Zygmund operator, briefly a Calderon- Zygmund operator, i. e., a linear operator bounded from $L_2(\mathbb{R}^n)$ in $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to the functions $Tf \in L_1^{loc}(\mathbb{R}^n)$ represented by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \text{ a. e. on supp} f.$$

Here $K(x; y)$ is a continuous function away from the diagonal which satisfies the standard estimates: there exist $c_1 > 0$ and $0 < \varepsilon \le 1$ such that

$$|K(x;y)| \le c_1 |x-y|^{-n}$$

for all $x, y \in \mathbb{R}^n$; $x \ne y$, and

$$|K(x;y) - K(x';y)| + |K(y;x) - K(y;x')| \le c_1 \left(\frac{|x-x'|}{x-y}\right)^{\varepsilon} |x-y|^{-n},$$

whenever $2|x - x'| \le |x - y|$. Such operators were introduced in [6].

The operators M and T play an important role in real and harmonic analysis and applications (see, for example [26] and [27]).

Theorem 2.1. *[8] Let $\Omega \subset \mathbf{R}^n$ be an open bounded set and $p \in WL(\Omega)$ satisfy condition (2.1). Then the maximal operator M is bounded in $L^{p(\cdot)}(\Omega)$.*

Singular operators within the framework of the spaces with variable exponents were studied in [9]. From Theorem 4.8 and Remark 4.6 of [9] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω, but valid for an arbitrary open set Ω under the corresponding condition in $p(x)$ at infinity.

Theorem 2.2. *[9] Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and $p \in WL(\Omega)$ satisfy condition (2.1). Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.*

3. $L^{p(\cdot)}$ a priori estimates

We consider the Dirichlet problem (1.2) in bounded domains Ω. From now on we will assume that $\partial\Omega$ is of class C^2. The solution of this problem is given by

$$u(x) = \int_\Omega G(x, y) f(y)dy \tag{3.1}$$

where $G(x, y)$ is the Green function, which can be written as

$$G(x, y) = \Gamma(x - y) + h(x; y)$$

with $h(x, y)$ satisfying, for each fixed $y \in \Omega$,

$$\begin{cases} \Delta_x h(x, y) = 0 & x \in \Omega \\ h(x, y) = -\Gamma(x - y) & x \in \partial\Omega. \end{cases}$$

If $P(y,Q)$ is the Poisson kernel, $h(x, y)$ is given by

$$h(x; y) = -\frac{1}{(n-2)\omega_n}\int_{\partial\Omega}\frac{1}{|x-Q|^{n-2}}P(y,Q)dS(Q),$$

where dS denotes the surface measure on $\partial\Omega$.

In what follows the letter C will denote a generic constant, not necessarily the same at each occurrence. It is known that the Green function satisfies the following estimates (see [29]),

$$G(x; y) \le \begin{cases} C\log|x-y| & if \ \ n = 2, \\ C|x-y|^{2-n} & if \ \ n \ge 3, \end{cases}$$

and

$$|D_{x_i} G(x, y)| \le C|x - y|^{1-n}.$$

Therefore

$$D_{x_i} u(x) = \int_\Omega D_{x_i} G(x,y) f(y) dy.$$

To obtain the second derivatives of u from the representation (3.1) we will use the following lemma. We denote with $d(x)$ the distance to the boundary, namely,

$$d(x) = \inf_{Q \in \partial\Omega} |x - Q| .$$

Lemma 3.1. *Given* $\alpha \in \mathbf{Z}_+^n$ $(|\alpha| > 0$ *if* $n = 2)$ *there exists a constant C depending only on n and* α *such that*

$$|D^\alpha h(x, y)| \le C d(x)^{2-n-|\alpha|} .$$

It follows from this lemma that for each $x \in \Omega$, $D_{x_i x_j} h(x,y)$ is bounded uniformly in a neighborhood of x and so

$$D_{x_i x_j} \int_\Omega h(x, y) f(y)\, dy = \int_\Omega D_{x_i x_j} h(x, y) f(y)\, dy$$

On the other hand, since $|D_{x_j} \Gamma(x)| \le C|x|^{1-n}$ we have

$$D_{x_j} \int_\Omega \Gamma(x- y) f(y)\, dy = \int_\Omega D_{x_j} \Gamma(x- y) f(y)\, dy$$

However, $D_{x_i x_j} \Gamma$ is not an integrable function and we cannot interchange the order between second derivatives and integration. A known standard argument shows that

$$D_{x_i} \int_\Omega D_{x_j} \Gamma(x- y) f(y)\, dy = K f(x) + c(x) f(x)$$

where c is a bounded function and

$$Kf(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} D_{x_i x_j} \Gamma(x - y) f(y) dy.$$

Here and in what follows we consider f defined in $\mathbf{R}^n$ extending the original f by zero.

The operator K is a Calder´on-Zygmund singular integral operator. Indeed, since $D_{x_j} \Gamma \in C^\infty (\mathbf{R}^n \setminus \{0\})$ and it is a homogeneous function of degree $1 - n$, it follows that $D_{x_i x_j} \Gamma(x - y)$ is homogeneous of degree $-n$ and has vanishing average on the unit sphere (see Lemma 11.1 in [2], page 152). Then, it follows from the general theory given in [5] that K is a bounded operator in $L^{p(\cdot)}(\mathbf{R}^n)$ for $p \in WL(\Omega)$ satisfy condition (2.1).

Moreover, the maximal operator

$$\widetilde{K} f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y|>\varepsilon} D_{x_i x_j} \Gamma(x - y) f(y) dy \right|$$

is also bounded in $L^{p(\cdot)}(\mathbf{R}^n)$ for $p \in WL(\Omega)$ satisfy condition (2.1).

We can now state and prove our main result.

Theorem 3.1. *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded* C^2 *domain. If* $p \in WL(\Omega)$ *satisfy condition (2.1) and u is the solution of problem (1.2), then there exists a constant C depending only on n ,* $p(\cdot)$ *and* Ω *such that*

$$\|u\|_{W^{2,p(\cdot)}(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}, \tag{3.2}$$

Proof. We will need the following estimate for the Green function. This estimate has been proved by A. Dall'Acqua and G. Sweers in [7], however they assume that the domain is more regular than C^2. Let Ω be a bounded C^2 domain and $G(x; y)$ be the Green function of problem (1.2) in Ω. There exists a constant C depending only on n and Ω such that for $(x; y) \in \Omega\times\Omega$

$$|D_{x_i x_j} G(x; y)| \leq C\frac{d(x)}{|x-y|^{n+1}}.$$

Our result follows from the following inequalities (see [11]).

There exists a constant C depending only on n and Ω such that, for any $x \in \Omega$,

$$|u(x)| + |D_{x_i} u(x)| \leq CMf(x), \tag{3.3}$$

$$|D_{x_i x_j} u(x)| \leq C(\widetilde{K} f(x) + Mf(x) + |f(x)|). \tag{3.4}$$

Theorems 2.1 and 2.2 imply that the operators M and $\widetilde{K}$ are bounded in $L^{p(\cdot)}(\Omega)$. Therefore (3.2) follows immediately from inequalities (3.3) and (3.4).

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PUASSON TƏNLİYİ ÜÇÜN $L^{p(\cdot)}$ APRİOR QİYMƏTLƏNDİRMƏLƏR

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XÜLASƏ

Fərz edək ki, $\Omega\ R^n$ -də məhdud oblastdır, $\partial\Omega \subset C^2$ və u klassik Puasson məsələsinin Ω -də həllidir, yəni

$$-\Delta u = f, \quad \Omega,$$
$$u = 0, \qquad \partial\Omega,$$

burada $f \in L^{p(\cdot)}(\Omega)$, $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$.
Bu məqalədə əsas nəticə aşağıdakı aprior qiymətləndirmənin alınmasıdır:

$$\|u\|_{W^{2,p(\cdot)}(\Omega)} \le C\|f\|_{L^{p(\cdot)}(\Omega)},$$

Açar sözlər: Puasson tənliyi, Qrin funksiyası, Kalderon-Ziqmund nəzəriyyəsi, dəyişən dərəcəli Lebeq fəzaları, dəyişən dərəcəli Sobolev fəzaları.

АПРИОРНЫЕ ОЦЕНКИ В $L^{p(\cdot)}$ ДЛЯ УРАВНЕНИЯ ПУАССОНА

С.С.АЛИЕВ

РЕЗЮМЕ

Пусть Ω- ограниченная область в R^n с границей $\partial\Omega \subset C^2$ и u решение классической задачи Пуассона в Ω, т.е.

$$-\Delta u = f, \quad \Omega,$$
$$u = 0, \qquad \partial\Omega,$$

где $f \in L^{p(\cdot)}(\Omega)$, $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$.
Основной результат задачи-доказательство следующей априорной оценки:

$$\|u\|_{W^{2,p(\cdot)}(\Omega)} \le C\|f\|_{L^{p(\cdot)}(\Omega)}.$$

Ключевые слова: уравнение Пуассона, функция Грина, теория Калдерона-Зигмунда, пространства Лебега с меняющимися показателями, пространства Соболева с меняющимися показателями.

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